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US ARMY MEDICAL RESEARCH LABORATORY

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REPORT NO. 710

THE CHI SQUARE TEST OF GOODNESS OF FIT: EXACT CRITICAL
VALUES FOR THE CASE OF EQUIPROBABLE ALTERNATIVES

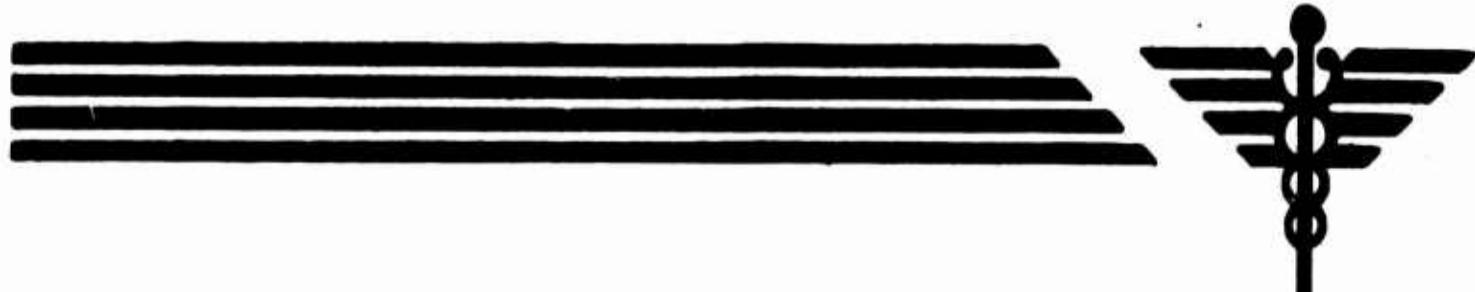
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Progress Report
by
James N. Cronholm, M. S.

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Acknowledgments

The computer program for the calculations on which the tables in this report are based was written by John C. Hoff. Carl E. Guthrie helped compile Table 1.

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AD

REPORT NO. 710

**THE CHI SQUARE TEST OF GOODNESS OF FIT: EXACT CRITICAL
VALUES FOR THE CASE OF EQUIPROBABLE ALTERNATIVES**

Progress Report

by

James N. Cronholm, M. S.

**Experimental Psychology Division
US ARMY MEDICAL RESEARCH LABORATORY
Fort Knox, Kentucky 40121**

5 December 1966

**Audition and Auditory Perception
Work Unit No. 030
Basic Research in Performance Effectiveness
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ABSTRACT

THE CHI SQUARE TEST OF GOODNESS OF FIT: EXACT CRITICAL VALUES FOR THE CASE OF EQUIPROBABLE ALTERNATIVES

OBJECTIVE

To present a table of exact critical values of the chi square criterion of goodness of fit for the case of equiprobable alternatives, and to compare the exact test with the more familiar continuous approximation.

RESULTS

Exact critical values are given for the chi square criterion for the equiprobable case when there are n observations and k categories ($2 \leq n \leq 15$, $2 \leq k \leq 15$). Nominal significance levels given are: $\alpha \leq .200$, $.100$, $.050$, $.025$, $.010$, $.005$, $.001$. Exact significance levels are presented for $\alpha \leq .050$, and it is shown that the true significance level of the approximation with $\alpha = .050$ exceeds $.050$ in 27 percent of the cases examined.

CONCLUSIONS

Statistical tests of the hypothesis of equal category probabilities are conservative with respect to Type I errors if the proposed tables are used. The standard chi square test of this hypothesis is not uniformly conservative.

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THE CHI SQUARE TEST OF GOODNESS OF FIT: EXACT CRITICAL VALUES FOR THE CASE OF EQUIPROBABLE ALTERNATIVES

When n independent observations are distributed over k mutually exclusive and exhaustive categories, the chi square criterion, given by

$$(1) \quad \chi^2 = \sum_{i=1}^k \frac{(n_i - np_i)^2}{np_i},$$

is frequently used as a measure of the goodness of fit of observed frequencies, n_i , to expected frequencies, np_i . In the special case considered here, the population category probabilities are assumed to be equal, $p_i = 1/k$ ($i = 1, 2, \dots, k$). The purpose of this paper is to present a table of exact critical values of χ^2 and compare the exact test with its more familiar continuous approximation.

Table 1 gives exact critical values of χ^2 for $2 \leq n \leq 15$, $2 \leq k \leq 15$, and selected significance levels, $\alpha \leq .200, .100, .050, .025, .010, .005, .001$. The entries in Table 1 were extracted from recently prepared tables of exact cumulative sampling probabilities produced by computer expansion of a generating function [3]. While it is possible to obtain the required sampling distributions by hand, the computations are very lengthy even when n and k are relatively small: the 196 tables of cumulative sampling probabilities from which Table 1 was derived required approximately 15 minutes of IBM 7094 computing time.

Some parts of Table 1 can be obtained from earlier work. Cartwright [1] tabulated critical values of his index of agreement, Alpha, for $3 \leq n \leq 5$, $4 \leq k \leq 16$. It can easily be shown that $\text{Alpha} = (\chi^2 + n - k)/k(n - 1)$ when the $p_i = 1/k$ ($i = 1, 2, \dots, k$). However, Cartwright's critical values were obtained from interpolation between points on the uncumulated distributions. Cronholm [2, p. 27] tabulated uncumulated sampling probabilities of Alpha for $2 \leq n \leq 13$, $2 \leq k \leq 11$. But these probabilities are only given to four places beyond the decimal point and their cumulation is probably inaccurate [2, p. 31]. Various authors have also calculated exact sampling distributions of χ^2 for particular values of n and k . For example, Walker and Lev [6] consider the two cases $k = 3$, $n = 6, 12$ in their introduction to the continuous approximation. Other examples include [4, 5, 7, 8].

To test an hypothesis of equal category probabilities, Table 1 is entered at α , n , and k . If the observed value of the criterion (1) is

equal to or greater than the value tabulated, the hypothesis may be rejected at or beyond the level specified by α . Because the exact sampling distribution of χ^2 is discrete and rather irregular with respect to both χ^2 and its sampling probabilities, Table 1 gives the smallest obtainable value of χ^2 with a cumulative probability of $1 - \alpha$ or more.

As an example, suppose that $n = 5$ observations are distributed over $k = 10$ categories and the value of χ^2 is found to be 29.0000. To test the hypothesis that the sample came from a population with equal category probabilities, $p_i = 1/10$ ($i = 1, 2, \dots, 10$), enter Table 1 at $n = 5$, $k = 10$, and $\alpha = .05$, say. The critical value is found to be 21.0000, and since this is less than the observed value of 29.0000, the hypothesis may be rejected. The observed value is also significant at the .005 level. Notice that the exact test does not require that expected frequencies, n/k , be greater than some minimum - usually 5. In the present example they are equal to .5. Thus, the problems of pooling categories or discarding data do not arise.

Table 2 gives the exact significance levels of the corresponding entries in Table 1 at the nominal .050 level. All of the actual significance levels are less than .050, but some are substantially less, e.g., $n = 4$, $k = 8$. The average exact value of α is .0316, so that the exact test is somewhat conservative.

Table 3 has been prepared in order to make certain comparisons between the exact test of Table 1 and the standard χ^2 distribution. The entries in Table 3 are the exact significance levels of the smallest observable values of χ^2 which yield significance at the .050 level using the continuous approximation. In other words, these entries are the true significance levels operating when the standard test is used with $\alpha = .050$. If all the entries in Table 3 were equal to the corresponding entries in Table 2, the two tests would be equivalent with respect to Type I errors over the range of n and k covered.

A comparison of Tables 2 and 3 will show that this is not generally the case. Two entries in Table 3 ($n = 12$, $k = 7$; $n = 15$, $k = 5$) are less than the corresponding entries in Table 2, 141 are equal (including the blanks where no obtainable value of χ^2 yields significance), and 53 (underlined) entries in Table 3 exceed .050. While the average true significance level for the approximate test is .0431, in 53 cases significance will be claimed at the .050 level when the true value of α is greater than .050. Unfortunately, these 53 cases are scattered rather uniformly over the table so that no simple rule involving n and k can insure a conservative test using the approximation. However,

the most serious errors occur when n is small, e.g., $n = 2$, $k = 9$, 10, 11, and $n = 3$, $k = 3$: as a compromise, if $n > 6$, the true significance levels will be no greater than .0704 ($n = 12$, $k = 3$) when the approximate test is used with $\alpha = .050$. This will be true regardless of the size of the expected frequencies, provided they are equal and $n, k \leq 15$. A better rule is to use Table 1.

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TABLE I

Critical values of χ^2 for $2 \leq k \leq 15$, $2 \leq n \leq 15$, and $p_i = 1/k$ $\alpha = .200$

| $n \setminus k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|--------|--------|---------|---------|---------|---------|---------|
| 2 | - | - | - | 8.0000 | 10.0000 | 12.0000 | 14.0000 |
| 3 | - | 6.0000 | 9.0000 | 12.0000 | 15.0000 | 18.0000 | 21.0000 |
| 4 | 4.0000 | 8.0000 | 12.0000 | 8.5000 | 8.0000 | 10.0000 | 12.0000 |
| 5 | 5.0000 | 5.2000 | 5.4000 | 8.0000 | 10.6000 | 10.4000 | 12.6000 |
| 6 | 6.0000 | 4.0000 | 6.0000 | 9.0000 | 12.0000 | 10.3333 | 12.6667 |
| 7 | 3.5714 | 4.5714 | 7.2357 | 8.0000 | 7.5714 | 10.0000 | 12.4286 |
| 8 | 4.5000 | 4.0000 | 6.0000 | 7.0000 | 8.5000 | 9.5000 | 12.0000 |
| 9 | 2.7778 | 4.6667 | 5.6667 | 7.1111 | 7.6667 | 10.4444 | 11.4444 |
| 10 | 3.6000 | 3.8000 | 5.2000 | 7.0000 | 8.0000 | 9.6000 | 10.8000 |
| 11 | 4.4545 | 4.5455 | 5.3636 | 6.7273 | 8.0909 | 10.0000 | 11.5455 |
| 12 | 3.0000 | 4.5000 | 5.3333 | 6.3333 | 8.0000 | 9.0000 | 10.6667 |
| 13 | 3.7692 | 3.8462 | 5.1538 | 6.6154 | 7.7692 | 9.0769 | 11.0000 |
| 14 | 2.5714 | 4.0000 | 5.4286 | 6.7143 | 8.2857 | 9.0000 | 11.1429 |
| 15 | 3.2667 | 3.6000 | 5.0000 | 6.6667 | 7.8000 | 9.7333 | 11.1333 |

| $n \setminus k$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 2 | 16.0000 | 18.0000 | 20.0000 | 22.0000 | 24.0000 | 26.0000 | 28.0000 |
| 3 | 24.0000 | 27.0000 | 30.0000 | 33.0000 | 36.0000 | 39.0000 | 22.0000 |
| 4 | 14.0000 | 16.0000 | 18.0000 | 20.0000 | 22.0000 | 24.0000 | 26.0000 |
| 5 | 14.8000 | 13.0000 | 14.8000 | 16.6000 | 18.4000 | 20.2000 | 22.0000 |
| 6 | 15.0000 | 14.0000 | 16.0000 | 18.0000 | 20.0000 | 22.0000 | 24.0000 |
| 7 | 12.2857 | 14.4286 | 16.5714 | 18.7143 | 17.1429 | 19.0000 | 20.8571 |
| 8 | 12.2500 | 14.5000 | 16.7500 | 16.0000 | 18.0000 | 20.0000 | 22.0000 |
| 9 | 12.0000 | 14.3333 | 14.2222 | 16.3333 | 18.4444 | 20.5556 | 22.6667 |
| 10 | 13.4000 | 14.0000 | 14.2000 | 16.4000 | 18.6000 | 18.0000 | 20.0000 |
| 11 | 12.7273 | 13.5455 | 16.0000 | 16.2727 | 18.5455 | 18.2727 | 20.3636 |
| 12 | 12.0000 | 13.0000 | 15.5000 | 16.0000 | 18.3333 | 18.3333 | 20.5000 |
| 13 | 12.6154 | 13.9231 | 14.9231 | 15.6154 | 18.0000 | 18.2308 | 20.4615 |
| 14 | 11.7143 | 13.1429 | 14.2857 | 15.1429 | 17.5714 | 18.0000 | 20.2857 |
| 15 | 12.0000 | 13.6667 | 15.0667 | 16.2000 | 17.0667 | 17.6667 | 20.0000 |

TABLE I (Continued)

Critical values of χ^2 for $2 \leq k \leq 15$, $2 \leq n \leq 15$, and $p_i = 1/k$

$\alpha = .100$

| $n \setminus k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|--------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | - | 9.0000 | 12.0000 | 15.0000 | 18.0000 | 21.0000 |
| 4 | - | 8.0000 | 12.0000 | 16.0000 | 11.0000 | 13.5000 | 12.0000 |
| 5 | 5.0000 | 10.0000 | 8.6000 | 8.0000 | 10.6000 | 13.2000 | 15.8000 |
| 6 | 6.0000 | 7.0000 | 7.3333 | 9.0000 | 12.0000 | 15.0000 | 18.0000 |
| 7 | 7.0000 | 5.4286 | 7.2857 | 10.8571 | 11.0000 | 12.0000 | 12.4286 |
| 8 | 4.5000 | 6.2500 | 7.0000 | 8.2500 | 10.0000 | 13.0000 | 14.0000 |
| 9 | 5.4444 | 6.0000 | 6.5556 | 8.2222 | 10.3333 | 12.0000 | 13.2222 |
| 10 | 6.4000 | 5.0000 | 7.6000 | 8.0000 | 10.4000 | 12.4000 | 12.4000 |
| 11 | 4.4545 | 5.0909 | 6.8182 | 7.6364 | 10.2727 | 11.2727 | 13.0000 |
| 12 | 5.3333 | 6.0000 | 6.6667 | 8.0000 | 10.0000 | 11.3333 | 13.3333 |
| 13 | 3.7692 | 5.6923 | 7.0000 | 8.1538 | 9.6154 | 11.2308 | 12.2308 |
| 14 | 4.5714 | 5.2857 | 7.1429 | 8.1429 | 10.0000 | 11.0000 | 12.2857 |
| 15 | 5.4000 | 5.2000 | 6.6000 | 8.0000 | 9.4000 | 11.6000 | 13.2667 |

| $n \setminus k$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 2 | - | 18.0000 | 20.0000 | 22.0000 | 24.0000 | 26.0000 | 28.0000 |
| 3 | 24.0000 | 27.0000 | 30.0000 | 33.0000 | 36.0000 | 39.0000 | 42.0000 |
| 4 | 14.0000 | 16.0000 | 18.0000 | 20.0000 | 22.0000 | 24.0000 | 26.0000 |
| 5 | 18.4000 | 17.0000 | 19.2000 | 21.4000 | 23.6000 | 25.8000 | 22.0000 |
| 6 | 15.0000 | 17.3333 | 19.6667 | 22.0000 | 24.3333 | 22.0000 | 24.0000 |
| 7 | 14.8571 | 17.2857 | 19.7143 | 18.7143 | 20.8571 | 23.0000 | 25.1429 |
| 8 | 14.5000 | 17.0000 | 16.7500 | 19.0000 | 21.2500 | 23.5000 | 25.7500 |
| 9 | 16.0000 | 16.5556 | 16.6667 | 19.0000 | 21.3333 | 20.5556 | 22.6667 |
| 10 | 15.2000 | 16.0000 | 18.6000 | 18.8000 | 21.2000 | 20.8000 | 23.0000 |
| 11 | 14.3636 | 15.3636 | 18.0000 | 18.4545 | 20.9091 | 23.3636 | 23.0909 |
| 12 | 15.0000 | 16.3333 | 17.3333 | 18.0000 | 20.5000 | 20.6667 | 23.0000 |
| 13 | 14.0000 | 15.4615 | 16.6154 | 19.3077 | 20.0000 | 20.3846 | 22.7692 |
| 14 | 14.2857 | 16.0000 | 17.4286 | 18.5714 | 19.4286 | 22.0000 | 22.4286 |
| 15 | 14.4000 | 15.0000 | 16.5333 | 17.8000 | 20.5333 | 21.4000 | 22.0000 |

TABLE I (Continued)

Critical values of χ^2 for $2 \leq k \leq 15$, $2 \leq n \leq 15$, and $p_i = 1/k$ $\alpha = .050$

| $n \setminus k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|--------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | - | - | 12.0000 | 15.0000 | 18.0000 | 21.0000 |
| 4 | - | 8.0000 | 12.0000 | 16.0000 | 20.0000 | 24.0000 | 28.0000 |
| 5 | - | 10.0000 | 15.0000 | 12.0000 | 15.4000 | 13.2000 | 15.8000 |
| 6 | 6.0000 | 12.0000 | 11.3333 | 10.6667 | 14.0000 | 15.0000 | 18.0000 |
| 7 | 7.0000 | 8.8571 | 9.5714 | 10.8571 | 14.4286 | 14.0000 | 17.0000 |
| 8 | 8.0000 | 7.0000 | 8.0000 | 10.7500 | 11.5000 | 13.0000 | 16.0000 |
| 9 | 5.4444 | 8.0000 | 8.3333 | 10.4444 | 11.6667 | 13.5556 | 15.0000 |
| 10 | 6.4000 | 7.4000 | 8.4000 | 10.0000 | 11.6000 | 13.8000 | 15.6000 |
| 11 | 7.3636 | 6.7273 | 8.2727 | 10.3636 | 11.3636 | 13.8182 | 15.9091 |
| 12 | 5.3333 | 6.5000 | 8.0000 | 9.6667 | 12.0000 | 12.5000 | 14.6667 |
| 13 | 6.2308 | 6.6154 | 8.2308 | 9.6923 | 11.4615 | 13.3846 | 14.6923 |
| 14 | 7.1429 | 7.0000 | 8.2857 | 10.2857 | 11.7143 | 13.0000 | 14.5714 |
| 15 | 5.4000 | 6.4000 | 8.2000 | 9.3333 | 11.8000 | 13.4667 | 14.3333 |

| $n \setminus k$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | 24.0000 | 27.0000 | 30.0000 | 33.0000 | 36.0000 | 39.0000 | 42.0000 |
| 4 | 18.5000 | 21.0000 | 23.5000 | 20.0000 | 22.0000 | 24.0000 | 26.0000 |
| 5 | 18.4000 | 21.0000 | 23.6000 | 26.2000 | 28.8000 | 25.8000 | 28.0000 |
| 6 | 21.0000 | 24.0000 | 19.6667 | 22.0000 | 24.3333 | 26.6667 | 29.0000 |
| 7 | 17.4286 | 17.2857 | 19.7143 | 22.1429 | 24.5714 | 27.0000 | 29.4286 |
| 8 | 16.7500 | 19.5000 | 22.2500 | 22.0000 | 21.2500 | 23.5000 | 25.7500 |
| 9 | 16.0000 | 18.7778 | 21.5556 | 21.6667 | 24.2222 | 23.6667 | 26.0000 |
| 10 | 17.0000 | 18.0000 | 18.6000 | 21.2000 | 23.8000 | 26.4000 | 26.0000 |
| 11 | 16.0000 | 19.0000 | 20.0000 | 20.6364 | 23.2727 | 23.3636 | 25.8182 |
| 12 | 16.5000 | 18.0000 | 19.1667 | 22.0000 | 22.6667 | 23.0000 | 25.5000 |
| 13 | 16.7692 | 18.5385 | 20.0000 | 21.1538 | 22.0000 | 24.6923 | 25.0769 |
| 14 | 16.8571 | 17.4286 | 19.0000 | 22.0000 | 23.1429 | 24.0000 | 24.5714 |
| 15 | 16.8000 | 17.6667 | 19.4667 | 21.0000 | 22.2667 | 23.2667 | 26.0000 |

TABLE I (Continued)

Critical values of χ^2 for $2 \leq k \leq 15$, $2 \leq n \leq 15$, and $p_i = 1/k$

$\alpha = .025$

| $n \setminus k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|--------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | - | - | - | - | 18.0000 | 21.0000 |
| 4 | - | - | 12.0000 | 16.0000 | 20.0000 | 24.0000 | 28.0000 |
| 5 | - | 10.0000 | 15.0000 | 20.0000 | 15.4000 | 18.8000 | 22.2000 |
| 6 | - | 12.0000 | 11.3333 | 15.6667 | 14.0000 | 17.3333 | 20.6667 |
| 7 | 7.0000 | 8.8571 | 9.5714 | 12.2857 | 14.4286 | 18.0000 | 17.0000 |
| 8 | 8.0000 | 10.7500 | 11.0000 | 12.0000 | 14.5000 | 16.5000 | 16.0000 |
| 9 | 9.0000 | 8.0000 | 11.0000 | 12.6667 | 13.0000 | 16.6667 | 16.7778 |
| 10 | 6.4000 | 7.4000 | 10.0000 | 12.0000 | 12.8000 | 15.2000 | 17.2000 |
| 11 | 7.3636 | 8.9091 | 9.7273 | 11.2727 | 13.5455 | 15.0909 | 17.3636 |
| 12 | 8.3333 | 8.0000 | 10.0000 | 11.3333 | 13.0000 | 16.0000 | 16.0000 |
| 13 | 6.2308 | 8.0000 | 10.0769 | 12.0000 | 13.3077 | 15.5385 | 17.1538 |
| 14 | 7.1429 | 7.4286 | 9.4286 | 12.4286 | 13.4286 | 15.0000 | 16.8571 |
| 15 | 8.0667 | 8.4000 | 9.2667 | 12.0000 | 13.4000 | 15.3333 | 16.4667 |

| $n \setminus k$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | 24.0000 | 27.0000 | 30.0000 | 33.0000 | 36.0000 | 39.0000 | 42.0000 |
| 4 | 32.0000 | 36.0000 | 40.0000 | 44.0000 | 28.5000 | 31.0000 | 33.5000 |
| 5 | 18.4000 | 21.0000 | 23.6000 | 26.2000 | 28.8000 | 31.4000 | 34.0000 |
| 6 | 21.0000 | 24.0000 | 27.0000 | 30.0000 | 33.0000 | 26.6667 | 29.0000 |
| 7 | 20.0000 | 23.0000 | 22.8571 | 22.1429 | 24.5714 | 27.0000 | 29.4286 |
| 8 | 19.0000 | 22.0000 | 22.2500 | 25.0000 | 27.7500 | 30.5000 | 29.5000 |
| 9 | 20.0000 | 21.0000 | 21.5556 | 24.3333 | 27.1111 | 26.7778 | 29.3333 |
| 10 | 18.8000 | 20.0000 | 23.0000 | 23.6000 | 23.8000 | 26.4000 | 29.0000 |
| 11 | 19.2727 | 20.8182 | 22.0000 | 25.0000 | 25.6364 | 25.9091 | 28.5455 |
| 12 | 18.0000 | 19.6667 | 22.8333 | 24.0000 | 24.8333 | 27.6667 | 28.0000 |
| 13 | 18.1538 | 20.0769 | 21.6923 | 23.0000 | 26.0000 | 26.8462 | 27.3846 |
| 14 | 18.1429 | 20.2857 | 22.1429 | 23.7143 | 25.0000 | 26.0000 | 28.8571 |
| 15 | 18.0000 | 20.3333 | 22.4000 | 22.6000 | 24.0000 | 27.0000 | 28.0000 |

TABLE I (Continued)

Critical values of χ^2 for $2 \leq k \leq 15$, $2 \leq n \leq 15$, and $p_i = 1/k$

$\alpha = .010$

| $n \setminus k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | - | - | - | - | - | - |
| 4 | - | - | - | 16.0000 | 20.0000 | 24.0000 | 28.0000 |
| 5 | - | - | 15.0000 | 20.0000 | 25.0000 | 30.0000 | 22.2000 |
| 6 | - | 12.0000 | 18.0000 | 15.6667 | 20.0000 | 17.3333 | 20.6667 |
| 7 | - | 14.0000 | 14.1429 | 13.7143 | 17.8571 | 18.0000 | 21.5714 |
| 8 | 8.0000 | 10.7500 | 12.0000 | 13.2500 | 16.0000 | 18.2500 | 20.0000 |
| 9 | 9.0000 | 12.6667 | 11.0000 | 16.0000 | 17.0000 | 16.6667 | 20.3333 |
| 10 | 10.0000 | 10.4000 | 13.2000 | 15.0000 | 16.4000 | 18.0000 | 18.8000 |
| 11 | 11.0000 | 8.9091 | 11.1818 | 14.0000 | 15.7273 | 17.6364 | 20.2727 |
| 12 | 8.3333 | 10.5000 | 12.0000 | 13.8333 | 16.0000 | 17.1667 | 20.0000 |
| 13 | 9.3077 | 9.3846 | 11.9231 | 13.5385 | 16.0769 | 17.6923 | 19.6154 |
| 14 | 10.2857 | 9.5714 | 12.2857 | 13.8571 | 15.1429 | 18.0000 | 19.1429 |
| 15 | 8.0667 | 10.0000 | 11.4000 | 13.3333 | 15.8000 | 17.2000 | 19.6667 |

| $n \setminus k$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | 27.0000 | 30.0000 | 33.0000 | 36.0000 | 39.0000 | 42.0000 |
| 4 | 32.0000 | 36.0000 | 40.0000 | 44.0000 | 48.0000 | 52.0000 | 56.0000 |
| 5 | 25.6000 | 29.0000 | 32.4000 | 26.2000 | 28.8000 | 31.4000 | 34.0000 |
| 6 | 24.0000 | 27.3333 | 30.6667 | 30.0000 | 33.0000 | 36.0000 | 39.0000 |
| 7 | 25.1429 | 23.0000 | 26.0000 | 29.0000 | 32.0000 | 35.0000 | 33.7143 |
| 8 | 21.2500 | 22.0000 | 25.0000 | 28.0000 | 31.0000 | 30.5000 | 33.2500 |
| 9 | 22.0000 | 23.2222 | 26.4444 | 27.0000 | 27.1111 | 29.8889 | 32.6667 |
| 10 | 22.4000 | 24.0000 | 25.2000 | 28.4000 | 29.0000 | 29.2000 | 32.0000 |
| 11 | 20.9091 | 22.6364 | 26.0000 | 27.1818 | 28.0000 | 31.0000 | 31.2727 |
| 12 | 22.5000 | 23.0000 | 24.6667 | 26.0000 | 29.1667 | 30.0000 | 33.0000 |
| 13 | 20.9231 | 23.1538 | 25.0769 | 26.6923 | 28.0000 | 31.1538 | 32.0000 |
| 14 | 20.7143 | 23.1429 | 25.2857 | 27.1429 | 28.7143 | 30.0000 | 31.0000 |
| 15 | 21.6000 | 23.0000 | 25.3333 | 25.8000 | 29.2000 | 30.7333 | 32.0000 |

TABLE I (Continued)

Critical values of χ^2 for $2 \leq k \leq 15$, $2 \leq n \leq 15$, and $p_i = 1/k$ $\alpha = .005$

| $n \setminus k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | - | - | - | - | - | - |
| 4 | - | - | - | - | 20.0000 | 24.0000 | 28.0000 |
| 5 | - | - | 15.0000 | 20.0000 | 25.0000 | 30.0000 | 35.0000 |
| 6 | - | 12.0000 | 18.0000 | 24.0000 | 20.0000 | 24.3333 | 20.6667 |
| 7 | - | 14.0000 | 21.0000 | 19.4286 | 17.8571 | 22.0000 | 23.8571 |
| 8 | - | 16.0000 | 17.0000 | 17.0000 | 16.0000 | 20.0000 | 22.0000 |
| 9 | 9.0000 | 12.6667 | 14.5556 | 16.0000 | 18.3333 | 21.3333 | 20.3333 |
| 10 | 10.0000 | 14.6000 | 13.2000 | 15.0000 | 16.4000 | 20.8000 | 22.0000 |
| 11 | 11.0000 | 11.6364 | 13.3636 | 14.9091 | 16.8182 | 18.9091 | 21.7273 |
| 12 | 12.0000 | 10.5000 | 14.6667 | 15.5000 | 18.0000 | 19.5000 | 21.3333 |
| 13 | 9.3077 | 11.2308 | 13.1538 | 15.8462 | 17.0000 | 19.8462 | 22.0769 |
| 14 | 10.2857 | 10.8571 | 12.8571 | 15.2857 | 16.8571 | 19.0000 | 21.4286 |
| 15 | 11.2667 | 11.2000 | 13.5333 | 15.3333 | 17.4000 | 19.0667 | 21.8000 |

| $n \setminus k$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | - | - | - | - | - | 42.0000 |
| 4 | 32.0000 | 36.0000 | 40.0000 | 44.0000 | 48.0000 | 52.0000 | 56.0000 |
| 5 | 40.0000 | 29.0000 | 32.4000 | 35.8000 | 39.2000 | 42.6000 | 34.0000 |
| 6 | 24.0000 | 27.3333 | 30.6667 | 34.0000 | 37.3333 | 40.6667 | 39.0000 |
| 7 | 25.1429 | 28.7143 | 32.2857 | 29.0000 | 32.0000 | 35.0000 | 38.0000 |
| 8 | 25.7500 | 27.0000 | 27.7500 | 28.0000 | 31.0000 | 34.0000 | 37.0000 |
| 9 | 24.0000 | 27.6667 | 28.8889 | 29.6667 | 32.8889 | 33.0000 | 32.6667 |
| 10 | 24.2000 | 26.0000 | 27.4000 | 30.8000 | 31.6000 | 34.8000 | 35.0000 |
| 11 | 24.1818 | 24.4545 | 28.0000 | 29.3636 | 30.3636 | 33.5455 | 36.7273 |
| 12 | 24.0000 | 26.3333 | 26.5000 | 28.0000 | 31.3333 | 32.3333 | 35.5000 |
| 13 | 23.6923 | 24.6923 | 28.4615 | 30.3846 | 30.0000 | 33.3077 | 34.3077 |
| 14 | 23.2857 | 24.5714 | 26.8571 | 28.8571 | 30.5714 | 32.0000 | 33.1429 |
| 15 | 24.0000 | 25.6667 | 26.8000 | 29.0000 | 30.9333 | 32.6000 | 34.0000 |

TABLE I (Continued)

Critical values of χ^2 for $2 \leq k \leq 15$, $2 \leq n \leq 15$, and $p_i = 1/k$ $\alpha = .001$

| $n \setminus k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | - | - | - | - | - | - |
| 4 | - | - | - | - | - | - | - |
| 5 | - | - | - | - | 25.0000 | 30.0000 | 35.0000 |
| 6 | - | - | 18.0000 | 24.0000 | 30.0000 | 36.0000 | 42.0000 |
| 7 | - | - | 21.0000 | 28.0000 | 24.7143 | 30.0000 | 26.1429 |
| 8 | - | 16.0000 | 24.0000 | 23.2500 | 22.0000 | 27.0000 | 24.0000 |
| 9 | - | 18.0000 | 19.8889 | 20.4444 | 21.0000 | 26.0000 | 27.4444 |
| 10 | - | 20.0000 | 17.2000 | 19.0000 | 22.4000 | 25.0000 | 25.2000 |
| 11 | 11.0000 | 16.5455 | 15.5455 | 20.3636 | 21.1818 | 25.2727 | 26.0909 |
| 12 | 12.0000 | 14.0000 | 18.0000 | 18.8333 | 22.0000 | 24.1667 | 26.6667 |
| 13 | 13.0000 | 15.3846 | 16.8462 | 19.6923 | 21.6154 | 24.1538 | 25.7692 |
| 14 | 14.0000 | 13.8571 | 16.8571 | 18.8571 | 21.1429 | 24.0000 | 26.0000 |
| 15 | 11.2667 | 14.8000 | 16.2000 | 18.6667 | 22.2000 | 22.8000 | 26.0667 |

| $n \setminus k$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | - | - | - | - | - | - |
| 4 | - | 36.0000 | 40.0000 | 44.0000 | 48.0000 | 52.0000 | 56.0000 |
| 5 | 40.0000 | 45.0000 | 50.0000 | 55.0000 | 60.0000 | 65.0000 | 70.0000 |
| 6 | 33.0000 | 37.3333 | 41.6667 | 34.0000 | 37.3333 | 40.6667 | 44.0000 |
| 7 | 30.2857 | 34.4286 | 38.5714 | 35.8571 | 39.4286 | 43.0000 | 46.5714 |
| 8 | 28.0000 | 32.0000 | 33.2500 | 37.0000 | 40.7500 | 44.5000 | 44.5000 |
| 9 | 32.0000 | 29.8889 | 31.3333 | 35.0000 | 38.6667 | 42.3333 | 42.6667 |
| 10 | 29.6000 | 32.0000 | 34.0000 | 35.6000 | 36.8000 | 37.6000 | 41.0000 |
| 11 | 27.4545 | 29.9091 | 34.0000 | 35.9091 | 37.4545 | 38.6364 | 42.1818 |
| 12 | 30.0000 | 29.6667 | 32.0000 | 36.0000 | 37.8333 | 41.6667 | 40.5000 |
| 13 | 29.2308 | 30.8462 | 31.8461 | 34.0769 | 36.0000 | 39.7692 | 41.2308 |
| 14 | 28.4286 | 30.2857 | 33.1429 | 35.7143 | 36.1429 | 38.0000 | 39.5714 |
| 15 | 27.6000 | 29.6667 | 32.6667 | 35.4000 | 37.8667 | 38.2000 | 40.0000 |

TABLE 2

**Exact Significance Levels of the Corresponding
Critical Values in Table 1**
 $\alpha = .050$

| $n \setminus k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|-------|-------|-------|-------|-------|-------|-------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | - | - | .0400 | .0278 | .0204 | .0156 |
| 4 | - | .0370 | .0156 | .0080 | .0046 | .0029 | .0020 |
| 5 | - | .0123 | .0039 | .0336 | .0201 | .0379 | .0259 |
| 6 | .0313 | .0041 | .0186 | .0272 | .0136 | .0379 | .0259 |
| 7 | .0156 | .0206 | .0208 | .0323 | .0158 | .0354 | .0219 |
| 8 | .0078 | .0334 | .0336 | .0366 | .0483 | .0341 | .0198 |
| 9 | .0391 | .0248 | .0457 | .0321 | .0484 | .0399 | .0403 |
| 10 | .0215 | .0224 | .0371 | .0398 | .0403 | .0364 | .0345 |
| 11 | .0117 | .0376 | .0448 | .0316 | .0407 | .0311 | .0275 |
| 12 | .0386 | .0485 | .0483 | .0395 | .0329 | .0491 | .0421 |
| 13 | .0225 | .0378 | .0379 | .0454 | .0386 | .0382 | .0435 |
| 14 | .0129 | .0331 | .0459 | .0336 | .0385 | .0433 | .0453 |
| 15 | .0352 | .0429 | .0376 | .0467 | .0418 | .0366 | .0493 |

| $n \setminus k$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------------|-------|-------|-------|-------|-------|-------|-------|
| 2 | - | - | - | - | - | - | - |
| 3 | .0123 | .0100 | .0083 | .0069 | .0059 | .0051 | .0044 |
| 4 | .0453 | .0370 | .0308 | .0451 | .0387 | .0335 | .0293 |
| 5 | .0184 | .0136 | .0103 | .0080 | .0063 | .0457 | .0401 |
| 6 | .0184 | .0136 | .0438 | .0345 | .0277 | .0225 | .0185 |
| 7 | .0437 | .0399 | .0299 | .0229 | .0180 | .0143 | .0116 |
| 8 | .0450 | .0311 | .0221 | .0394 | .0489 | .0388 | .0313 |
| 9 | .0400 | .0261 | .0176 | .0413 | .0312 | .0484 | .0395 |
| 10 | .0365 | .0395 | .0490 | .0346 | .0250 | .0184 | .0414 |
| 11 | .0469 | .0303 | .0368 | .0423 | .0305 | .0451 | .0342 |
| 12 | .0403 | .0431 | .0438 | .0300 | .0392 | .0479 | .0360 |
| 13 | .0335 | .0323 | .0355 | .0392 | .0445 | .0320 | .0432 |
| 14 | .0378 | .0456 | .0448 | .0284 | .0338 | .0385 | .0475 |
| 15 | .0359 | .0449 | .0381 | .0376 | .0405 | .0477 | .0338 |

TABLE 3

**Exact Significance Levels of the Smallest Observable
Values of χ^2 which Yield Significance at the .050
Level Using the Continuous Approximation**

| $n \setminus k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|-------|-------|-------|-------|-------|-------|-------|
| 2 | - | - | - | - | - | - | - |
| 3 | - | .1111 | .0625 | .0400 | .0278 | .0204 | .0156 |
| 4 | .1250 | .0370 | .0156 | .0080 | .0046 | .0729 | .0566 |
| 5 | .0625 | .0123 | .0625 | .0336 | .0201 | .0379 | .0259 |
| 6 | .0313 | .0535 | .0186 | .0272 | .0586 | .0379 | .0259 |
| 7 | .0156 | .0206 | .0515 | .0323 | .0158 | .0354 | .0611 |
| 8 | .0703 | .0590 | .0336 | .0538 | .0483 | .0341 | .0198 |
| 9 | .0391 | .0504 | .0457 | .0321 | .0484 | .0399 | .0403 |
| 10 | .0215 | .0590 | .0371 | .0398 | .0403 | .0364 | .0345 |
| 11 | .0654 | .0376 | .0448 | .0316 | .0407 | .0311 | .0528 |
| 12 | .0386 | .0704 | .0483 | .0395 | .0329 | .0395 | .0421 |
| 13 | .0225 | .0378 | .0379 | .0454 | .0386 | .0382 | .0435 |
| 14 | .0574 | .0582 | .0459 | .0519 | .0385 | .0433 | .0453 |
| 15 | .0352 | .0429 | .0376 | .0414 | .0418 | .0366 | .0493 |

| $n \setminus k$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------------|-------|-------|-------|-------|-------|-------|-------|
| 2 | .1111 | .1000 | .0909 | .0833 | .0769 | .0714 | .0667 |
| 3 | .0123 | .0100 | .0083 | .0069 | .0059 | .0051 | .0044 |
| 4 | .0453 | .0370 | .0308 | .0451 | .0387 | .0335 | .0293 |
| 5 | .0184 | .0856 | .0718 | .0611 | .0526 | .0457 | .0401 |
| 6 | .0184 | .0568 | .0438 | .0345 | .0277 | .0225 | .0797 |
| 7 | .0437 | .0399 | .0299 | .0229 | .0180 | .0622 | .0519 |
| 8 | .0450 | .0663 | .0506 | .0394 | .0489 | .0388 | .0313 |
| 9 | .0400 | .0261 | .0557 | .0413 | .0602 | .0484 | .0395 |
| 10 | .0365 | .0395 | .0490 | .0346 | .0673 | .0524 | .0414 |
| 11 | .0469 | .0548 | .0368 | .0423 | .0305 | .0451 | .0342 |
| 12 | .0403 | .0431 | .0438 | .0550 | .0392 | .0479 | .0360 |
| 13 | .0335 | .0571 | .0582 | .0392 | .0445 | .0579 | .0432 |
| 14 | .0514 | .0456 | .0448 | .0504 | .0542 | .0385 | .0475 |
| 15 | .0539 | .0449 | .0381 | .0376 | .0405 | .0477 | .0540 |

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13. ABSTRACT

A table of exact critical values of the chi square criterion of goodness of fit for n observations in k equiprobable categories is presented ($2 \leq n \leq 15$, $2 \leq k \leq 15$). The nominal significance levels are: $\alpha \leq .200, .100, .050, .025, .010, .005, .001$. Exact significance levels are presented for $\alpha \leq .050$, and it is shown that the true significance levels of the standard (continuous) chi square test with $\alpha = .050$ exceed .050 in 27 percent of the cases examined. Thus, while the exact test is uniformly conservative with respect to Type I errors, the standard test is not. (15)

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